

RANDOM COEFFICIENT MODELLING OF EXCHANGE RATE DYNAMICS

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INTRODUCTION

It is well known that the movements of exchange rate changes are heteroskedastic. The autoregressive conditional heteroskedasticity (ARCH) model introduced by Engle (1982) is a convenient way to formulate the conditional variances as a function of past observations. This class of models has been used by, for examples, Diebold and Nerlove (1988), Diebold and Pauly (1988), Domowitz and Hakkio (1985) and Hsieh (1987) to study the dynamics of foreign exchange rates.

Recently Tsay (1987) shows that the class of ARCH processes is a special case of the class of random coefficient autoregressive (RCA) processes, studied extensively by Nicholls and Quinn (1980, 1982) and Quinn and Nicholls (1981, 1982). The relationship between these two types of models can be illustrated by the following example. Consider a RCA(3) process,

$$X_t = \sum_{i=1}^3 (\beta_i + \eta_{it}) X_{t-i} + \varepsilon_t, \text{ where} \quad (1)$$

$$\begin{bmatrix} \eta_t \\ \varepsilon_t \end{bmatrix} \sim \text{iid} \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & \sigma^2 \end{bmatrix} \right], \text{ and,} \quad (2)$$

$$\eta_t = (\eta_{3t} \ \eta_{2t} \ \eta_{1t})'. \quad (3)$$

The conditional variance of this RCA(3) process is

$$V_{t-1}(X_t) = \sigma^2 + \sum_{i,j=1}^3 \omega_{ij} X_{t-i} X_{t-j}; \quad \Omega = (\omega_{ij})_{i,j=1,2,3}. \quad (4)$$

$V_t(\cdot)$ is the conditional variance operator based on the information available up to time t . Next, let us consider a ARCH(2,1) process

defined by

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t, \text{ where} \quad (5)$$

$$V_{t-1}(\varepsilon_t) = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2, \quad (6)$$

$$\alpha_i \geq 0 ; i = 0, 1.$$

It can be shown that the RCA(3) model is equivalent to the ARCH(2.1) model up to the second-order conditional moments if the following conditions are satisfied:

$$\beta_1 = \phi_1, \beta_2 = \phi_2, \beta_3 = 0;$$

$$\omega_{11} = \alpha_1, \omega_{12} = -\alpha_1 \phi_1, \omega_{13} = -\alpha_1 \phi_2, \omega_{22} = \alpha_1 \phi_1^2, \omega_{23} = \alpha_1 \phi_1 \phi_2,$$

$$\omega_{33} = \alpha_1 \phi_2^2; \text{ and } \sigma^2 = \alpha_0.$$

In general any given ARCH model is related to a RCA model via a set of non-linear constraints.⁽¹⁾

It is clear that an ARCH process is a more parsimonious way to study heteroskedasticity. However, as shown by Pagan and Sabau (1987), when a RCA process is mis-specified as an ARCH process, maximum likelihood method will give a biased estimator for the conditional mean. It is better to check for the presence of RCA effects before we estimate an ARCH specification unless we have a prior information to rule out the covariance terms (or to impose the stringent restrictions required to reduce a RCA process to an ARCH process; see footnote (1)). Moreover, the RCA specification allows both the magnitude and the sign of the past observations to affect the conditional variance. This feature may be useful for modelling financial prices whose volatility is related to both the size and the direction of price movements (see Nelson (1987),

Black (1976) and Christie (1982)).

The purpose of this paper is to investigate if the RCA model an appropriate choice to study the time series properties of changes in exchange rates. The plan of this paper is as follows. The procedures used to study uni-variate RCA models are introduced in section one. The empirical results are reported in section two. In section three the latent factor approach suggested by Diebold and Nerlove (1988) is proposed to estimate a multi-variate RCA model. The multi-variate model results are also reported in that section. The concluding remarks in section four ends the paper.

I: FITTING A UNIVARIATE RCA(P) MODEL.

Consider a time series record $\{X_1, \dots, X_T\}$. The p-th order random coefficient autoregressive process (RCA(p)) is given by

$$X_t = \sum_{i=1}^p (\beta_i + \eta_{it}) X_{t-i} + \varepsilon_t, \text{ where} \quad (7)$$

$$\begin{bmatrix} \eta_t \\ \varepsilon_t \end{bmatrix} \sim \text{iid} \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & \sigma^2 \end{bmatrix} \right], \text{ and,} \quad (8)$$

$$\eta_t = (\eta_{pt}, \dots, \eta_{1t})'. \quad (9)$$

Nicholls (1986) suggests a modified Box-Jenkins three-step modeling procedure for this class of time series model. In the first step we have to determine the order of the process using the AIC or SBC criterion and to test for the randomness of coefficients. It is well known that changes in exchange rates show little, if any, serial

correlations. Therefore more attention is put on the test for coefficient randomness. The data are assumed to follow a RCA(p) model if the test for coefficient randomness indicates that there is coefficient randomness in the first p-th order models but not in the (p+1)-th order model. In the second stage a maximum likelihood procedure is used to estimate the RCA(p) model. Finally, the standardized residuals are used to evaluate if the RCA(p) model adequately represents the data.

The $\bar{\tau}$ statistic derived in Quinn and Nicholls (1982) is used to test the null hypothesis that the data are generated by a fixed coefficient model. The statistic is defined as

$$\begin{aligned}\bar{\tau} &= \tau && \text{if } \hat{c} \text{ is (semi-)positive definite,} \\ &= 0 && \text{otherwise,}\end{aligned}$$

where $\tau = (2\nu\hat{\sigma}^4)^{-1}T\gamma'W\gamma$,

$$\hat{\varepsilon}_t = X_t - \sum_{i=1}^p \hat{\beta}_i X_{t-i},$$

$$\hat{\sigma}^2 = T^{-1} \sum_t \hat{\varepsilon}_t^2;$$

$\hat{\beta}$ is the estimate of β under the null,

$$\nu = (2T)^{-1} \sum_t \{(\hat{\varepsilon}_t^2 / \hat{\sigma}^2) - 1\}^2,$$

$$\hat{\gamma} = \left\{ \sum_t (z_t - \bar{z})(z_t - \bar{z})' \right\}^{-1} \left\{ \hat{\varepsilon}_t^2 (z_t - \bar{z}) \right\};$$

the least squares estimate of $\text{vech}(c)$,

$$W = T^{-1} \sum_t (z_t - \bar{z})(z_t - \bar{z})',$$

$$z_t = \text{Givec}((X_{t-p}, \dots, X_{t-1})(X_{t-p}, \dots, X_{t-1})'), \text{ and,}$$

$$\bar{z} = T^{-1} \sum_t z_t.$$

See Quinn and Nicholls (1980) for more detailed discussion on $\hat{\gamma}$ and the matrix G. The null hypothesis is rejected at the α % significance level if

$$\bar{r} > \chi^2_{(p+1)p/2, \alpha}.$$

The \bar{r} statistic has an intuitive interpretation. Re-arranging the terms, we can show that $T\bar{r}$ equals

$$\left\{ T^{-1} \sum_t \hat{\varepsilon}_t^2 (z_t - \bar{z}) \right\}' \left\{ \left[T^{-1} \sum_t (z_t - \bar{z})(z_t - \bar{z})' \right] \left[T^{-1} \sum_t (\hat{\varepsilon}_t^2 - \hat{\sigma}^2) \right] \right\}^{-1} \left\{ T^{-1} \sum_t \hat{\varepsilon}_t^2 (z_t - \bar{z}) \right\},$$

the sample covariance of z_t and $\hat{\varepsilon}_t^2$ weighted by the sample variances of z_t and $\hat{\varepsilon}_t^2$. It is obvious that \bar{r} is zero under the null hypothesis.

In this paper the parameters of the RCA(p) model are estimated by the maximum likelihood method. The likelihood of the model, L, is given by

$$L = (2\pi)^{-T/2} \prod_t \{ h_t^{c.5} \exp[-(X_t - \sum_{i=1}^p \beta_i X_{t-i})^2 / (2h_t)] \}, \quad (10)$$

where h_t is the conditional variance and is equal to $\sigma^2 + \gamma z_t$. Quinn and Nicholls (1981) shows that the maximum likelihood estimators obtained from maximizing equation (10) are strongly consistently and satisfy a central limit theorem. The maximum likelihood estimates are computed by the Davidon-Fletcher-Powell (DFP) algorithm. Non-negativity constraints are explicitly imposed to ensure both the variances and conditional variance are positive.

Finally, the Box-Pierce portmanteau Q-statistics computed from the standardized residuals and the squares of these residuals are used to evaluate the adequacy of the estimated RCA models. The standardized

residual is defined as

$$r_t = (X_t - \sum_{i=1}^p \hat{\beta}_i X_{t-i}) / \hat{h}_t, \quad (11)$$

where the "hat" denotes maximum likelihood estimates. The Q-statistic computed from r_t 's is used to detect serial correlations not captured by the estimated model. The Q-statistic calculated from r_t^2 's is used to test for serial correlations in the second moment or, equivalently, the heteroskedasticity not explained by the estimated model (see McLeod and Li (1983)).

II: EMPIRICAL RESULTS OF THE UNIVARIATE RCA MODELS

This section reports the results of fitting uni-variate RCA models to changes in exchange rates. The end-of-week US Dollar/British Pound (BP), US Dollar/Deutsche Mark (DM) and US Dollar/Swiss Franc (SF) exchange rates are used. The data are transformed by taking the first logarithmic differences and multiplied by 1000.⁽²⁾

The \bar{r} -statistics (not reported) suggest that both BP and SF follow RCA(3) processes and DM follows a RCA(2) process. The covariance estimates obtained from the higher order processes were non-positive definite. However, the Box-Pierce Q-statistic computed from the squares of the standardized residuals indicated that the RCA(2) model does not explain the conditional heteroskedasticity of DM satisfactory. Then a RCA(3) model is fitted. It is found that the RCA(3) model passed the diagnostic tests. Therefore a RCA(3) model for each of these three exchange rates are reported.

The maximum likelihood estimates are reported in Table 1. In these three cases the covariance terms are usually significantly different from zero. The terms in the covariance matrix are large compared to the fixed autoregressive coefficients. Hence the observed exchange rate changes are expected to show more wild fluctuations than a fixed coefficient stationary time series model. This also may explain why it is so difficult to detect serial correlations in changes in exchange rates under a constant coefficient time series framework. Moreover the estimated variance-covariance terms indicated that there is a good chance for a large (small) change to be followed by another large (small) change, a phenomenon observed by Cornell and Dietrich (1978).

The sample Box-Pierce Q-statistics are also reported in Table 1. $Q_1(10)$ and $Q_1(20)$ are the respectively the 10-lag and 20-lag Q-statistics computed from the standardized residuals while $Q_2(10)$ and $Q_2(20)$ are the Q-statistics computed from the squares of the standardized residuals. These sample Q-statistics are smaller than the conventional critical values. The estimated models provide a reasonable description on the patterns of the serial correlation and heteroskedasticity.

III: A LATENT FACTOR MULTI-VARIATE RCA MODEL

In this section we motivate and estimate the multivariate RCA model with a factor structure. Since the data we studied are exchange rates against the US Dollar, the movements in these series are likely to be

correlated. For instance, this happens when the market is reacting to the news about the US economy. The conditional covariances, like the conditional variances of the individual series, may be time-varying. On the other hand there are news that are specific to individual countries. These country specific news make each exchange rate different from the others. The latent factor structure suggested in Diebold and Nerlove (1988) provides a convenient way to study this situation.⁽⁹⁾ It assumes that changes in exchange rates are made up of two components. One is the common factor which exhibit conditional heteroskedasticity. This common factor captures both the time-varying conditional variances and conditional covariances. The second component is the countries specific component which represents the shocks "unique" to individual countries.

Another advantage of the latent factor approach is the huge reduction in the number of parameters in the problem. Consider a general tri-variate RCA(3) model:

$$X_t = \sum_{i=1}^3 (\beta_i + \eta_{ti}) X_{t-i} + \varepsilon_t, \quad (12)$$

$$U = E(\varepsilon_t \varepsilon_t'),$$

$$\eta = (\eta_{t3} \ \eta_{t2} \ \eta_{t1}),$$

$$C = E(\eta \otimes \eta),$$

ε_t and η_{ti} are uncorrelated.

X_t is a 3x1 vector, β_i is a 3x3 matrix of fixed coefficients, η_{ti} is a 3x3 matrix of random coefficients, ε_t is a 3x1 error vector, U is a 3x3 error covariance matrix and C is a 9x9 covariance matrix of the random

coefficients. The total number of parameters is 411. It is easily seen that the number of parameters explodes quickly either the number of random variables or the number of lags involved. This makes the estimation of such models extremely costly if not impossible. However, when we impose a latent factor structure, we can reduce the number of parameters dramatically. Consider the following multivariate latent factor model;

$$X_t = \lambda F_t + \varepsilon_t, \quad (13)$$

$$F_t = \alpha F_{t-1} + V_t, \text{ where} \quad (14)$$

$$\lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ \lambda_2 & 0 & 0 \\ \lambda_3 & 0 & 0 \end{bmatrix},$$

$$F_t = (f_t \ f_{t-1} \ f_{t-2})',$$

$$\varepsilon_t = (\varepsilon_{1t} \ \varepsilon_{2t} \ \varepsilon_{3t}),$$

$$\text{Var}(\varepsilon_t) = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_1^2 \end{bmatrix}$$

$$\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$V_t = (\nu_t \ 0 \ 0)', \text{ and}$$

$$\begin{aligned} \text{Var}(\nu_t | I_{t-1}) = & \nu_0 + \nu_{11} f_{t-1}^2 + \nu_{22} f_{t-2}^2 + \nu_{33} f_{t-3}^2 + \\ & \nu_{12} f_{t-1} f_{t-2} + \nu_{12} f_{t-1} f_{t-2} + \nu_{13} f_{t-2} f_{t-3}. \end{aligned}$$

$$\begin{aligned} h_t = \text{Var}(X_t | I_{t-1}) &= \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} [\lambda_1 \ \lambda_2 \ \lambda_3] \text{Var}(f_t | I_{t-1}) + \text{Var}(\varepsilon_t). \\ &= \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} [\lambda_1 \ \lambda_2 \ \lambda_3] \text{Var}(\nu_t | I_{t-1}) + \text{Var}(\varepsilon_t). \end{aligned}$$

f_t is the unobserved component that affects all exchange rates . It is assumed to be uncorrelated with the individual error ε_{ti} . In equation (14) f_t is assumed to follow a RCA(3) model. h_t is the conditional covariance matrix of X_t . This covariance matrix is time-varying because the conditional variance of f_t is not constant over time. The number of parameters in this multivariate RCA(3) model is 13.

It is seen that the multivariate latent factor RCA model has a sound economic interpretation and is computationally less demanding. We regard this modeling strategy, at least, is a practical way to describe the interaction of the exchange rate series.

It is found from the univariate analysis that the fixed autoregressive coefficients in these three exchange rate series are similar to each other. Therefore the common factor is assumed to follow a RCA(3) process and equations (13) and (14) is used to model the joint dynamic properties of the three exchange rate series.

The log likelihood function of the multivariate latent factor RCA model is

$$\mathcal{L} = \frac{-3T}{2} \ln 2\pi - \frac{1}{2} \sum_t \ln |h_t| - \frac{1}{2} \sum_t (X_t - E_{t-1}(X_t))' h_t^{-1} (X_t - E_{t-1}(X_t)),$$

where E_{t-1} is the conditional expectations operator based on information up to time $t-1$. The maximum likelihood estimates are obtained by using the DFP algorithm⁽⁴⁾ to maximize the likelihood under the constraint $\lambda_1 + \lambda_2 + \lambda_3 = 1$. The constraint is need to ensure the parameters are identified.⁽⁵⁾ Non-negativity constraints are met by performing the maximization with respect to the standard deviations instead of the

variances. The maximization is performed sequentially. The parameters are divided into two subsets. The first subset θ_1 has λ_i and σ_i , $i=1,2,3$, as its elements and the second subset θ_2 has the rest of the parameters as its elements. We first set $\theta_1 = \hat{\theta}_1^0$ and maximize the likelihood with respect to θ_2 and obtain the estimate $\hat{\theta}_2^1$. Then we set $\theta_2 = \hat{\theta}_2^1$ and maximize the function with respect to θ_1 and obtain $\hat{\theta}_1^1$. These procedures are repeated until the parameter values converged.

The maximum likelihood estimation results are reported in Table 2. It is interesting to note that the third lagged autoregressive coefficient estimate, same as those in the univariate cases, is not significant. The estimated variance-covariance terms (ν_i 's) are similar to that of the DM. One of the covariance term is highly significant. The maximized log likelihood is -7988 which is larger than the sum of the individual maximized log likelihoods (-8554). This increase in likelihood indicates there is a substantial gain in information when the behavior of these exchange rate series is jointly modelled.

IV: CONCLUDING REMARKS

The heteroskedasticity in exchange rate changes is examined with a general class of stochastic processes: the random coefficient autoregressive (RCA) processes. The results suggest that the class of RCA models is a reasonable alternative to study the time-varying volatility of financial price series. The individual series of exchange rate changes is explained adequately by a RCA(3) model. The joint

behavior of the three exchange rate series is modelled by a multi-variate latent factor RCA model with the common factor follows a RCA(3) process.

It is known that an ARCH model is a special case of an RCA model. The RCA model is a more flexible way to model conditional heteroskedasticity. The price for such flexibility is the increase in the number of parameters to be estimated. However, mis-specifying a RCA model as an ARCH model will result biased estimators. Therefore it is advisable to check for coefficient randomness before we specify an ARCH model for conditional variances.

FOOTNOTES

- (1) See Theorem 1 in Tsay (1987) for a formal statement of the relationship between a given ARCH model and the corresponding RCA model.
- (2) The data were scaled up to avoid underflow when the multi-variate model is estimated.
- (3) Bollerslev (1987) suggests an alternative approach to model the multi-variate GARCH model which assumes the correlations are constant over time.
- (4) The Kalman filter technique is used to compute the likelihood function. See Diebold and Nerlove (1988) and Nerlove et al (1988) for more details on the computational aspects of estimating the multivariate latent factor model. Diebold and Nerlove (1988) discusses the possible improvements that can be made on this modelling strategy.
- (5) Diebold and Nerlove (1988) constrains the variance of f_t equals to 1 for identification purposes. However, this constraint is more difficult to enforce in this type of Kalman filter recursive estimation procedure.

REFERENCES

- Black, F., 1976, Studies of stock market volatility changes, Proceedings of the American Statistical Association, Business and Economic Statistics Section, 177-81.
- Christie, A.A., 1982, The stochastic behavior of common stock variances, Journal of Financial Economics, 7, 407-32.
- Cornell, W.B. and J.K. Dietrich, 1978, The efficiency of the market for foreign exchange under floating exchange rates, Review of Economics and Statistics, 111-20.
- Diebold, F. and P. Pauly, 1987, Endogenous risk in a portfolio balance rational expectations model of the Deutsche Mark-Dollar rate, European Economics Review, forthcoming.
- Diebold, F. and M. Nerlove, 1988, The dynamics of exchange rate volatility: A multivariate latent factor ARCH model, mimeo.
- Domowitz, I. and C.S. Hakkio, 1985, Conditional variance and the risk premium in the foreign exchange market, Journal of International Economics, 47-66.
- Engle, R.F., 1982, Autoregressive conditional heteroskedasticity with estimation of the variance of UK inflation, Econometrica, 50, 987-1007.
- Hsieh, D. A., 1988, The statistical properties of daily foreign exchange rates: 1974-1983, Journal of International Economics, 24, 129-45.
- McLeod, A.I. and W.K. Li, 1983, Diagnostic checking ARMA time series models using squared-residual autocorrelations, Journal of Time series, 4, 269-73.
- Mussa, M., 1979, Empirical regularities in the behavior of exchange

- rates and theories of the foreign exchange market, in K. Brunner and A. Meltzer ed., Policies for employment, prices and exchange rates, Carnegie-Rochester Conference Series on Public Policy, 11, 9-57.
- Nelson, D.B., 1987, conditional heteroskedasticity in asset returns: A new approach, M.I.T. mimeo.
- Nerlove, M., F. Diebold, H. van Beeck and Y.W. Cheung, 1988, A multivariate ARCH model of foreign exchange rate determination, mimeo.
- Nicholls, D.F., The Box-Jenkins approach to random coefficient autoregressive modelling, in J. Gani and M.B. Priestley ed., Essays in time series and the allied process, Journal of Applied Probability, special volume 23A, 231-40.
- Nicholls D.F. and B.G. Quinn, 1982, Random coefficient autoregressive models: An introduction, Springer-Verlag.
- Nicholls D.F. and B.G. Quinn, 1980, The estimation of random coefficient autoregressive model I, Journal of Time Series Analysis, 37-46.
- Pagan A.R. and H. Sabau, 1987, On the inconsistency of the MLE in certain heteroskedastic regression model, Working Paper No. 104, University of Rochester.
- Quinn B.G. and D.F. Nicholls, 1981, The estimation of random coefficient autoregressive model II, Journal of Time Series Analysis, 185-203.
- Quinn B.G. and D.F. Nicholls, 1982, Testing for the randomness of autoregressive coefficients, Journal of Time Series Analysis, 123-35.
- Tsay, R.S., 1987, Conditional heteroscedastic time series models, Journal of the American Statistical Association, 82, 590-604.

Table 1. Results of Univariate RCA Models

	BP	DM	SF
β_1	0.0841 (2.49)	0.1233 (4.04)	0.0844 (2.74)
β_2	0.0997 (2.96)	0.1516 (4.88)	0.1155 (3.92)
β_3	0.0321 (0.93)	-.0207 (-.66)	-.0211 (-.66)
Ω_{11}	0.2189 (4.92)	0.1322 (3.75)	0.1894 (5.50)
Ω_{12}	0.0192 (0.57)	0.1478 (6.07)	0.1213 (4.74)
Ω_{13}	-.1179 (-3.02)	0.0064 (0.26)	0.0243 (0.74)
Ω_{22}	0.2394 (5.11)	0.1936 (4.48)	0.1411 (4.27)
Ω_{23}	0.0859 (2.40)	0.0379 (1.34)	0.1186 (3.65)
Ω_{33}	0.3318 (5.89)	0.2055 (5.44)	0.3118 (7.45)
σ^2	75.181 (4.92)	102.76 (13.1)	121.84 (11.7)
L	-2799	-2824	-2931
$Q_1(10)$	11.6	6.83	3.66
$Q_1(20)$	15.1	15.5	14.6
$Q_2(10)$	6.21	10.6	8.78
$Q_2(20)$	14.6	26.7	25.8

Table 2: Results of the Multi-variate Factor RCA Model

λ_1	λ_2	λ_3	σ_1	σ_2	σ_3
0.2372 (5.25)	0.3582 (19.2)	0.4046 ----	10.691 (10.7)	4.3963 (4.40)	6.8275 (6.83)

α_1	α_2	α_3
0.1203 (3.57)	0.1482 (4.55)	-.0243 (-.72)

ν_{11}	ν_{12}	ν_{13}	ν_{22}	ν_{23}	ν_{33}	ν_o
0.2197 (9.06)	0.1376 (4.60)	0.0149 (0.45)	0.1794 (7.41)	0.0453 (1.34)	0.2659 (12.3)	618.23 (20.00)